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Note

On the Validity of Vortex-Tangle Simulations of Homogeneous Superfluid Turbulence

It has been known for decades [1] that the turbulent state of superfluid ^{4}He consists of a random, self-sustaining tangle of quantized vortex lines. In the last few years we have developed a theoretical treatment of this dynamical state which exploits the well-documented fact [2] that quantized vortices behave like classical ideal-fluid vortex filaments of fixed circulation $\kappa = h/m_4$ and a microscopic core radius of order 10^{-8} cm. Because of the very small core size, the quantized vortex filaments can be treated in the local induction approximation. The novel feature of our model has been the realization that nonlocal effects must be included to the extent that vortices must be allowed to reconnect to other vortices or to boundaries when they approach them closely enough. The reconnection process allows vortex filaments to multiply, and provides the key non-linear dynamical mechanism which sustains the chaotic vortex-tangle state. Scaling arguments and numerical simulations based on this model have accurately predicted every major property of the fully developed superfluid turbulent state. Aside from the original short letter [3] reporting some of the main results, our work has been described in two long papers, the first [4] devoted to the behavior of individually interacting lines, and the second [5] to the fully developed, homogeneous vortex-tangle state.

A recent paper [6] in this journal has questioned the validity of our calculations. Using the same model, but a different numerical algorithm for stepping vortex lines forward in time, the author finds that he does not obtain a self-sustaining, homogeneous vortex-tangle state. This appears to represent a clear-cut disagreement with our findings that needs to be explained. Unfortunately, the author of [6] has made a further observation which has served to obscure rather than clarify the issue. Specifically, when vortex lines are reconnected in any such simulation, more or less sharply cusped vortex configurations are created, depending on how the reconnections are made. If the spacing of the points specifying the vortex line is not kept small, these cusped regions are not adequately modelled. The typical consequence (see Fig. 7 of [6]) is that spurious vortex loops are generated at the cusps, the effect being to artificially add new vortex singularities to the simulation. The author of [6] observes that when this is happening in his computations, he does obtain a homogeneous vortex tangle, the properties of which are similar to those obtained by us. Not unnaturally, he infers from this that the results reported by us are also the consequence of this kind of malfunction, arising from the use of excessively large mesh spacing in our calculations. The difference in the algorithms

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used to step the vortex lines forward does not seem to be an issue, since he obtains the same results using either algorithm.

The purpose of this brief Letter is to sort out some of the confusion which now exists. First we point out that the inference drawn by the author of [6] about our calculations is quite incorrect—the homogeneous vortex-tangle we obtain is *not* the result of spurious vortex looping, which does not occur in our calculations. Once this is understood, it is necessary to look for the *real* reason why, with both vortex-stepping algorithms working properly, the author of [6] obtains results which differ so drastically from ours. The explanation will be seen to lie in a certain additional feature of our calculation which, regrettably, we did not spell out clearly in our initial short report and of which the author of [6] was not aware when he first did his work. Since the relevant issues have been discussed at some length in [5] we will keep the discussion here very brief.

We first address the notion that the homogeneous vortex tangle obtained in our calculations is due to spurious vortex looping. We have established by a detailed and extended inspection of our simulations that in fact such a malfunction never occurs, even if the mesh spacing is made larger than we would normally use. The reason for the difference between this behavior and the copious artificial vortex looping reported in [6] becomes obvious when one looks at Fig. 7 of [6]. It is clear that the author makes his reconnections so as to produce very sharp cusps in his vortex configuration. Not surprisingly, both his and our vortex-stepping algorithm have trouble modelling the subsequent motion properly, and artificial vortex loops are generated. In our calculations, on the other hand, vortex reconnections are made in a much less singular manner (see, for example, Fig. 1 of [3]) and the mesh spacing is always kept small compared to the smallest radius of curvature of the line. Thus we have in fact never experienced any difficulty in handling the subsequent motion properly.

The way in which we make our reconnections was not chosen arbitrarily, nor should it be viewed as representing any kind of artificial smoothing. As discussed in considerable detail in [4], two quantized vortices which approach each other to within some critical distance Δ will undergo a strong non-local interaction which brings the two lines together at a point. If the lines are then reconnected at the microscopic level, forming a sharp cusp, they rapidly separate as the cusp smoothes out. Our extensive, fully nonlocal calculations of this process (see Figs. 16–19 of [4]) led us to conclude that "the final stage of the process looks very much as though a crude macroscopic reconnection had occurred at the distance Δ." This is the reason for making the reconnections in our simulations in a smooth way when the lines are still a distance Δ apart. As discussed in [5], the way in which one chooses to make the reconnections, in fact, has virtually no effect on the computations, *provided*, of course, that numerical artifacts of the kind occurring in [6] are avoided.

We now consider the real reason for the different results obtained in the two calculations. The existence of the self-sustaining vortex tangle within the context of the local approximation depends on a balance between the line amplification process which takes place by the outward ballooning of vortex lines in the plane perpendicular to the driving velocity and a certain level of self-consistently maintained three-dimensional behavior caused by the vortex-vortex reconnections. If the simulated vortex tangle happens to fluctuate into a configuration which is not sufficiently three-dimensional, it will degenerate into a collection of non-interacting lines, never again establishing three-dimensional behavior. Although this turns out not to be a problem when realistic boundary conditions are used at the channel walls, it invariably occurs when periodic boundary conditions are applied to all faces of the computational volume. As discussed in [5], we ascribe this to the fact that in this artificial situation, vortex–vortex reconnections lead to the creation of infinite "open orbit" lines, which (unlike the large closed loops they replace) have a strong tendency to evolve into non-interacting straight lines. In those of our simulations which utilize periodic boundary conditions (including those reported in [3]) we have dealt with this problem by adding an occasional randomizing step in which some of the infinite lines are rotated by 90°. The aim of this admittedly heuristic procedure is to rescue the simulated vortex tangle from the kind of fluctuations which will take it out of the homogeneous state. The calculations described in [6], which are done with periodic boundary conditions only, differ from ours in not using such procedure. Hence they do not result in a homogeneous vortex tangle. The author of [6] interprets this failure to obtain a homogeneous vortex tangle as showing the inadequacy of the underlying model. We interpret it as an artifact arising from the use of periodic boundary conditions.

It is, of course, imperative to investigate whether the heuristic randomizing procedure employed in our calculations is merely (as we claim) a stratagem for circumventing an artificial problem introduced by the use of periodic boundary conditions, or whether it represents some kind of forced randomization which needs to be justified on physical grounds, e.g., as arising from long-range nonlocal effects neglected in our model. In investigating this question, we have found that when the use of periodic boundary conditions is avoided, a homogeneous, self-sustaining vortex tangle is obtained without difficulty and, in particular, without the use of the heuristic randomizing procedure. Extensive comparisons of calculations using real-wall boundary conditions without this procedure with those using periodic boundary conditions plus randomization show no significant differences in the vortex tangle properties obtained. We conclude that the difficulty of maintaining a homogeneous vortex tangle when using periodic boundary conditions is indeed an artifial phenomenon and that the randomizing procedure which we introduced in order to get sensible results does not represent an essential modification of the underlying model. It seems clear that it should rather be viewed as only a way of simulating the homogeneous state more efficiently, and it can be avoided entirely by not using periodic boundary conditions in the first place.

It appears from [6] that artificial vortex looping can also restore three-dimensional vortex-tangle behavior. While this has caused the confusion discussed above, it is not particularly surprising, since as such loops grow they will map into the computational volume as randomly oriented line segments (see Fig. 8 of [5]). Thus a steady supply of artificially generated vortex loops can also serve to at least crudely counteract the problems arising from periodic boundary conditions and (one error cancelling another) give the illusion of reproducing the results of our calculations. The results obtained in this way are by no means reliable; nevertheless, they are likely to be approximately correct, since the artificial addition of a few extra vortex loops is not going to change the properties of the vortex tangle dramatically.

In summary, it is certainly not the case that the results we have obtained in the past are due to spurious computational effects of the kind claimed in [6]. They occur there only because of the highly singular way in which the vortex–vortex reconnections are modelled. On the other hand, the calculations we reported in [3] differed from those of [6] in that they incorporated a randomizing procedure designed to counter certain artificial features arising from the use of periodic boundary conditions in the calculation. The calculations reported in [6] do not result in a homogeneous tangle because they do not incorporate such a feature. We have since found that if the use of periodic boundary conditions is avoided, none of these problems arise, and yet our earlier results are reproduced completely. We conclude that the failure to obtain a homogeneous vortex tangle reported in [6] most likely arises from the author's implementation of periodic boundary conditions (which differs from ours) and does not reflect a failure of the physical model. While there are many fascinating aspects of this problem which remain unexplored, there do not at present seem to be any substantial reasons to doubt the successes achieved so far.

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